Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Practice exam 3

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

(a) (i) We have $|-1+i\sqrt{3}| = 2$ and $\operatorname{Arg}(-1+i\sqrt{3}) = 2\pi/3$, so

$$(-1+i\sqrt{3})^4 = (2e^{2i\pi/3})^4 = 16e^{8i\pi/3} = 16e^{2i\pi/3}.$$ 3

(ii) We have $\operatorname{Log}(-1) = \log 1 + i\operatorname{Arg}(-1) = i\pi$. So

$$(-1)^{3i} = e^{3i\operatorname{Log}(-1)} = e^{-3\pi}.$$ 3

(b) Since $1+i = \sqrt{2}e^{i\pi/4}$, it follows that

$$w = \frac{1}{1+i} = \frac{1}{\sqrt{2}}e^{-i\pi/4}.$$

Hence

$$w^{1/4} = \frac{1}{2^{1/8}}e^{-i\pi/16}.$$ 4

10 Total

1.1

Question 2

(a) The function

$$f(z) = \frac{\sin z}{z - \pi}$$

is analytic on $\mathbb{C} - \{\pi\}$ and has a singularity at π. Observe that

$$\lim_{z \to \pi} (z - \pi) f(z) = \lim_{z \to \pi} \sin z = \sin \pi = 0.$$

Hence f has a removable singularity at π, by HB B4 3.1, p58. 3

(b) The function

$$f(z) = \frac{e^z - 1}{z^2}$$

is analytic on $\mathbb{C} - \{0\}$ and has a singularity at 0. Notice that

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots,$$

for $z \neq 0$. Let

$$g(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots.$$

The radius of convergence of this power series is ∞, so g is an entire function. Observe that

$$f(z) = \frac{g(z)}{z}, \quad \text{for } z \neq 0,$$

and $g(0) = 1$, so f has a pole of order 1 at 0, by HB B4 1.7, p55. 4

(c) The function

$$f(z) = \cosh \frac{1}{z}$$

is analytic on $\mathbb{C} - \{0\}$ and has a singularity at 0. We know that

$$\cosh w = 1 + \frac{w^2}{2!} + \frac{w^4}{4!} + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Substituting $w = 1/z$ gives

$$\cosh \frac{1}{z} = 1 + \frac{1}{2! z^2} + \frac{1}{4! z^4} + \cdots, \quad \text{for } z \neq 0.$$

This is the Laurent series about 0 for f. It has infinitely many non-zero terms in its singular part, so f has an essential singularity at 0, by HB B4 2.10(c), p57. 3

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Question 3

(a) The function f has a simple pole at each of the three solutions 1, $e^{2i\pi/3}$ and $e^{4i\pi/3}$ of $z^3 - 1 = 0$.

To simplify notation, we write $w = e^{2i\pi/3}$. Observe that $w^2 = e^{4i\pi/3}$, so the three solutions are 1, w and w^2.

We can calculate the residue at each pole using the g/h Rule with $g(z) = 1$ and $h(z) = z^3 - 1$, observing that $h'(z) = 3z^2$ is non-zero at each of the three poles of f. Using the fact that $w^3 = 1$, we obtain

$$\operatorname{Res}(f, 1) = \frac{g(1)}{h'(1)} = \frac{1}{3},$$

$$\operatorname{Res}(f, w) = \frac{g(w)}{h'(w)} = \frac{1}{3w^2} = \frac{w}{3},$$

$$\operatorname{Res}(f, w^2) = \frac{g(w^2)}{h'(w^2)} = \frac{1}{3w^4} = \frac{w^2}{3}.$$ 4

(b) The function f is analytic on the simply connected region $\{z : \operatorname{Re} z > -\frac{1}{2}\}$ apart from a simple pole at 1. This region contains Γ, and the point 1 lies inside Γ. Applying the Residue Theorem with one of the residues found in part (a), we obtain

$$\int_\Gamma \frac{1}{z^3 - 1}\, dz = 2\pi i \times \frac{1}{3} = \frac{2\pi i}{3}.$$ 2

(c) Let $p(t) = 1$ and $q(t) = t^3 - 1$. Then the degree of q exceeds that of p by $3 - 0 = 3$ and, by part (a), the poles of $f = p/q$ on the real axis are simple. Hence we can apply HB C1 3.8, p62, to see that

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\, dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of f at the poles in the upper half-plane, and T is the sum of the residues of f at the poles on the real axis. Using the residues found in part (a) we see that

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\, dt = 2\pi i \times \frac{w}{3} + \pi i \times \frac{1}{3} = \frac{\pi i}{3}(2w + 1).$$

Let's not leave the answer in that form; we should be sure that it is a real number. Since $w = e^{2\pi i/3} = -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, we see that $2w + 1 = i\sqrt{3}$. Hence

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\, dt = -\frac{\pi}{\sqrt{3}}.$$ 4

10 Total

Question 4

(a) We have

$$|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x. \qquad 2$$

(b) We have

$$|\sinh z| = \left|\tfrac{1}{2}(e^z - e^{-z})\right| \le \tfrac{1}{2}(|e^z| + |e^{-z}|) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x. \qquad 3$$

(c) Let

$$\phi_n(z) = \frac{\sinh z}{n^2+1},$$

for $n = 1, 2, \ldots$, and let $E = \{z : |\mathrm{Re}\, z| \le 3\}$. Using part (b) we see that if $z \in E$, then

$$|\phi_n(z)| \le \frac{\cosh x}{n^2+1} \le \frac{\cosh 3}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since

$$\sum_{n=1}^{\infty} \frac{\cosh 3}{n^2} = \cosh 3 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, by HB B3 1.9, p47, we see that

$$\sum_{n=1}^{\infty} \phi_n(z) = \sum_{n=1}^{\infty} \frac{\sinh z}{n^2+1}$$

is uniformly convergent on E, by the M-test. 5

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Question 5

(a) The conjugate velocity function

$$\overline{q}(z) = z + i$$

is entire, so q is the velocity function for an ideal flow on the whole of the complex plane $\mathbb{C}$, by HB D1 1.15, p81. 1

(b) The only solution of $q(z) = 0$ is $z = -i$. This is the unique stagnation point of the flow. 1

(c) A complex potential function for the flow is

$$\Omega(z) = \tfrac{1}{2}z^2 + iz,$$

since this function is a primitive of $\overline{q}$ on $\mathbb{C}$. Writing $z = x + iy$, we see that

$$\Omega(z) = \tfrac{1}{2}(x + iy)^2 + i(x + iy) = \tfrac{1}{2}(x^2 + 2ixy - y^2) + (ix - y).$$

Hence a stream function for the flow is

$$\Psi(z) = \operatorname{Im}\Omega(z) = xy + x.$$

The streamlines are given by $\Psi(z) = k$, for real constants k. The streamline through the point $1 = 1 + 0i$ satisfies

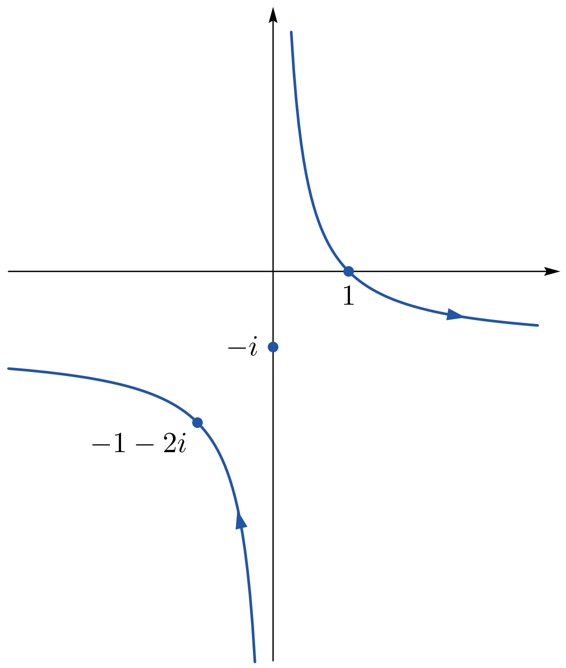
$$k = 0 + 1 = 1.$$

Hence an equation for this streamline is $xy + x = 1$. The streamline through the point $-1 - 2i$ satisfies

$$k = (-1) \times (-2) - 1 = 1.$$

Again, an equation for this streamline is $xy + x = 1$. 4

(d) At the point 1 we have $q(1) = 1 - i$ ('south east' flow) and at the point $-1 - 2i$ we have $q(-1 - 2i) = -1 + 2i - i = -1 + i$ ('north west' flow).



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Question 6

(a) (i) We have

$$f(i) = i^3 + i = -i + i = 0.$$

Next, we have

$$f(0) = 0^3 + i = i.$$

Hence $f^2(i) = f(f(i)) = i$, so i is a periodic point of period 2.
Observe that $f'(z) = 3z^2$. Hence the multiplier of f at i is

$$(f^2)'(i) = f'(i) \times f'(0) = 0.$$

Therefore i is a super-attracting periodic point of f. 4

(ii) Since f maps i to 0 and 0 to i, we see that 0 is also a periodic point of f. 2

(b) (i) According to HB D2 4.7(c), p92, the Mandelbrot set intersects the real axis in the interval $\left[-2, \frac{1}{4}\right]$. Hence $-\frac{3}{2} \in M$. 2

(ii) According to HB D2 4.7(a), p92, the Mandelbrot set is contained in $\{z : |z| \leq 2\}$. Now,

$$\left|-\tfrac{3}{2} + \tfrac{3}{2}i\right| = \tfrac{3}{2}|1+i| = \tfrac{3}{2} \times \sqrt{2} = \sqrt{\tfrac{9}{2}} > 2,$$

so $-\frac{3}{2} + \frac{3}{2}i \notin M$. 2

10 Total

Question 7

(a) (i) The set A is not a region because it is not open.

The set B is a region because it is open and connected.

The set $B - A$ is not a region because it is not connected (you cannot connect the points $\frac{1}{2}(1+i)$ and $2(1+i)$ in $B - A$ by a path in $B - A$). 3

(ii) The set A is compact because it is closed and bounded.

The set B is not compact because it is not bounded.

The set $B - A$ is not compact because it is not bounded. 3

(iii) First we prove that f is bounded on A.

The function f is continuous on $\mathbb{C} - \{0\}$. This set contains the set A. Hence f is continuous on the compact set A, so it is bounded on A by the Boundedness Theorem.

Next we prove that f is not bounded on B.

Let $z_n = \frac{1}{n}(1+i)$, for $n = 1, 2, \ldots$. Then $\operatorname{Arg} z_n = \pi/4$, so $z_n \in B$. We have

$$|f(z_n)| = \left|\frac{n}{1+i}\right| = \frac{n}{\sqrt{2}}.$$

Thus $|f(z_n)| \to \infty$ as $n \to \infty$, so f is not bounded on B. 4

(b) (i) Let $z = x + iy$. Then

$$\begin{aligned} f(z) = \overline{z}(1-z) = \overline{z} - |z|^2 &\\ &= (x - iy) - (x^2 + y^2) \\ &= (x - x^2 - y^2) - iy. \end{aligned}$$

1

(ii) Define

$$u(x, y) = x - x^2 - y^2 \quad \text{and} \quad v(x, y) = -y.$$

Then $f(z) = u(x, y) + iv(x, y)$, and

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= 1 - 2x, \\ \frac{\partial u}{\partial y}(x, y) &= -2y, \\ \frac{\partial v}{\partial x}(x, y) &= 0, \\ \frac{\partial v}{\partial y}(x, y) &= -1. \end{aligned}$$

The first Cauchy–Riemann equation is

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) &\iff 1 - 2x = -1 \\ &\iff x = 1. \end{aligned}$$

The second Cauchy–Riemann equation is

$$\begin{aligned} \frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) &\iff -2y = 0 \\ &\iff y = 0. \end{aligned}$$

Hence both the Cauchy–Riemann equations are satisfied if and only if $z = x + iy = 1 + 0i = 1$.

Since the partial derivatives exist and are continuous on $\mathbb{C}$, and the Cauchy–Riemann equations are satisfied at the points $z = 1$, we see from the Cauchy–Riemann Converse Theorem that f is differentiable at 1.

However, the Cauchy–Riemann equations are not satisfied at any other points, so the Cauchy–Riemann Theorem tells us that f is not differentiable at any points of $\mathbb{C} - \{1\}$. Hence f is not analytic at 1. 8

(iii) By the Cauchy–Riemann Converse Theorem,

$$f'(1) = \frac{\partial u}{\partial x}(1,0) + i\frac{\partial v}{\partial x}(1,0) = -1 + 0i = -1.$$ 1

20 Total

Question 8

(a) (i) We have

$$\cosh w = 1 + \frac{1}{2}w^2 + \frac{1}{24}w^4 + \cdots, \quad \text{for } w \in \mathbb{C},$$

$$\sinh z = z + \frac{1}{6}z^3 + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Let $w = \sinh z$. Since $\sinh 0 = 0$, we can apply the Composition Rule for Power Series to give

$$\begin{aligned}\cosh(\sinh z) &= 1 + \frac{1}{2}\left(z + \frac{1}{6}z^3 + \cdots\right)^2 + \frac{1}{24}(z + \cdots)^4 + \cdots \\ &= 1 + \frac{1}{2}\left(z^2 + \frac{1}{3}z^4 + \cdots\right) + \frac{1}{24}z^4 + \cdots \\ &= 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \cdots .\end{aligned}$$ 4

(ii) Since f is an entire function, this Taylor series converges to $f(z)$ for each $z \in \mathbb{C}$, by HB B3 3.5, p51. Hence the disc of convergence is $\mathbb{C}$. 2

(iii) The function $g(z) = z^3 f(1/z)$ is analytic on the simply connected region $\mathbb{C}$ except for a singularity at 0. By part (a)(i) we have

$$\begin{aligned}z^3 f(1/z) &= z^3\left(1 + \frac{1}{2}\frac{1}{z^2} + \frac{5}{24}\frac{1}{z^4} + \cdots\right) \\ &= z^3 + \frac{1}{2}z + \frac{5}{24z} + \ldots,\end{aligned}$$

for $z \in \mathbb{C} - \{0\}$. Hence

$$\operatorname{Res}(g, 0) = \frac{5}{24}.$$

Applying the Residue Theorem we see that

$$\int_C z^3 f(1/z)\, dz = 2\pi i \times \frac{5}{24} = \frac{5\pi i}{12}.$$ 4

(b) We have

$$g(z) = \frac{1}{z^2+1} = \frac{1}{z^2(1+1/z^2)}.$$

Let $w = 1/z^2$. Observe that $|z| > 1$ if and only if $|z|^2 > 1$ if and only if $|w| < 1$. Also,

$$\frac{1}{1+w} = 1 - w + w^2 - w^3 + \cdots, \quad \text{for } |w| < 1.$$

Hence

$$\begin{aligned} g(z) &= \frac{1}{z^2}\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \cdots\right) \\ &= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} + \cdots, \end{aligned}$$

for $|z| > 1$. 5

(c) (i) Let $z_n = e^{i/n}$, for $n = 1, 2, \ldots$.
Define

$$f(z) = (z - z_1)(z - z_2)\cdots(z - z_N).$$

This is a polynomial function, so it is entire, and it has degree N, so it is not constant. It satisfies $f(z_n) = 0$, for $n = 1, 2, \ldots, N$. 2

(ii) Suppose that g is an entire function that satisfies

$$g(z_n) = 0, \quad \text{for } n = 1, 2, \ldots.$$

Observe that $z_n \to 1$ as $n \to \infty$. Hence g is analytic on $\mathbb{C}$ and the set of zeros of g has a limit point in $\mathbb{C}$. By the Uniqueness Theorem, g is the zero function, a constant function.

Consequently, there are no non-constant functions g that satisfy $g(z_n) = 0$, for $n = 1, 2, \ldots$. 3

20 Total

Question 9

(a) Let $f(z) = \exp(z^3)$ and $\mathcal{R} = \{z : |z| < 3\}$. Then f is analytic on $\mathbb{C}$, so it is analytic (and non-constant) on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}} = \{z : |z| \leq 3\}$. We can therefore apply the Maximum Principle to see that the maximum value of $|f(z)|$ on $\overline{\mathcal{R}}$ is attained on the boundary $\partial\mathcal{R}$ and is not attained in $\mathcal{R}$. Hence

$$\max\{|f(z)| : |z| \leq 3\} = \max\{|f(z)| : |z| = 3\}.$$

Now, if $|z| = 3$, then $z = 3e^{it}$, where $0 \leq t < 2\pi$. Hence

$$\begin{aligned}|f(z)| &= |\exp(z^3)| \\ &= |\exp(27e^{3it})| \\ &= |\exp(27\cos 3t + 27i \sin 3t)| \\ &= \exp(27\cos 3t).\end{aligned}$$

Since $x \longmapsto e^x$ is an increasing real function, the expression $27\exp(\cos 3t)$ takes its maximum value when $\cos 3t = 1$. This happens when (and only when) $t = 0, 2\pi/3, 4\pi/3$, corresponding to the values

$$z = 3e^{i0} = 3, \quad z = 3e^{2\pi i/3} \quad \text{and} \quad z = 3e^{4\pi i/3}.$$

At these values,

$$|f(z)| = e^{27}.$$

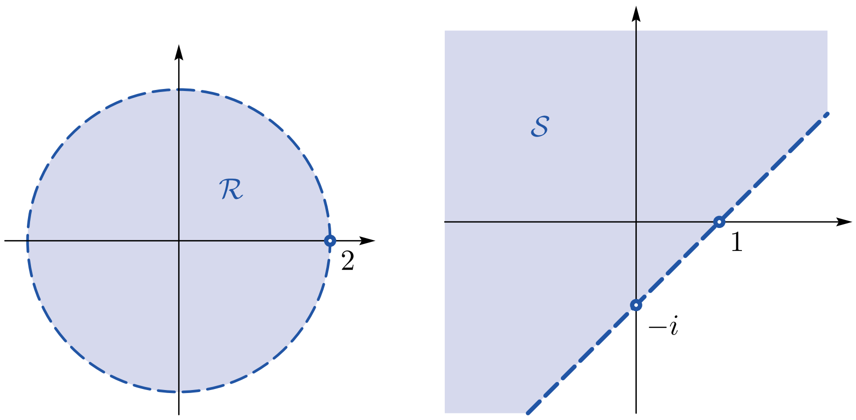
In summary, then,

$$\max\{|\exp(z^3)| : |z| \leq 3\} = e^{27},$$

and this maximum is attained at the points $z = 3, 3e^{2\pi i/3}, 3e^{4\pi i/3}$ only. 10

Remark: One can answer this question without using the Maximum Principle. To do this, write $z = re^{it}$, where $0 \leq r \leq 3$, and calculate $|f(z)|$.

(b) (i)



2

(ii) We choose a Möbius transformation f that maps three points on the boundary of $\mathcal{R}$ to three points on the boundary of $\mathcal{S}$. Let us choose f to satisfy

$$f(-2i) = -i, \quad f(2) = 1 \quad \text{and} \quad f(2i) = \infty.$$

Using the Implicit Formula for Möbius Transformations we have $w = f(z)$, where

$$\frac{z+2i}{z-2i}\frac{2-2i}{2+2i} = \frac{w+i}{w-\infty}\frac{1-\infty}{1+i}.$$

Simplifying this, we obtain

$$\frac{z+2i}{z-2i}\frac{1-i}{1+i} = \frac{w+i}{1+i},$$

so

$$\begin{aligned} w = (1-i)\frac{z+2i}{z-2i} - i &= \frac{(z+2i) - i(z+2i) - i(z-2i)}{z-2i} \\ &= \frac{(1-2i)z+2i}{z-2i}. \end{aligned}$$

That is,

$$f(z) = w = \frac{(1-2i)z+2i}{z-2i}.$$

By HB C3 4.4, p77, f maps $\mathcal{R}$ onto one of the two generalised open discs with boundary that of $\mathcal{S}$. Observe that $0 \in \mathcal{R}$ and $f(0) = -1 \in \mathcal{S}$. Hence f maps $\mathcal{R}$ onto $\mathcal{S}$, and because it is a Möbius transformation it is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$. 6

(iii) The mapping $z \longmapsto e^{i\theta}z$, where $\theta \in \mathbb{R}$, is an anticlockwise rotation about 0 through an angle θ. This is a one-to-one conformal mapping from $\mathcal{R}$ onto itself. Hence any mapping

$$g(z) = f(e^{i\theta}z)$$

is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, and there are infinitely many such mappings. 2

20 Total